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The Projection Approach to the Fokker–Planck Equation. I. Colored Gaussian Noise

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It is shown that the Fokker–Planck operator can be derived via a projection-perturbation approach, using the repartition of a more detailed operator $\mathscr{L}$ into a perturbation $\mathscr{L}_1$ and an unperturbed part $\mathscr{L}_0$. The standard Fokker–Planck structure is recovered at the second order in $\mathscr{L}_1$, whereas the perturbation terms of higher order are shown to provoke the breakdown of this structure. To get rid of these higher order terms, a key approximation, local linearization (LL), is made. In general, to evaluate at the second order in $\mathscr{L}_1$ the exact expression of the diffusion coefficient which simulates the influence of a Gaussian noise with a finite correlation time τ, a resummation up to infinite order in τ must be carried out, leading to what other authors call the best Fokker–Planck approximation (BFPA). It is shown that, due to the role of terms of higher order in $\mathscr{L}_1$, the BFPA leads to predictions on the equilibrium distributions that are reliable only up to the first order in τ. The LL, on the contrary, in addition to making the influence of terms of higher order in $\mathscr{L}_1$ vanish, results in a simple analytical expression for the term of second order that is formally coincident with the complete resummation over all the orders in τ provided by the Fox theory. The corresponding diffusion coefficient in turn is shown to lead in the limiting case $\tau \to \infty$ to exact results for the steady-state distributions. Therefore, over the whole range $0 \leqslant \tau \leqslant \infty$ the LL turns out to be an approximation much more accurate than the global linearization proposed by other authors for the same purpose of making the terms of higher order in $\mathscr{L}_1$ vanish. In the short-τ region the LL leads to results virtually coincident with those of the BFPA. In the large-τ region the LL is a more accurate approximation than the BFPA itself. These theoretical arguments are supported by the results of both analog and digital simulation.

KEY WORDS: Fokker–Planck approximation; colored noise; Gaussian noise; local linearization; global linearization.

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1. INTRODUCTION

The problem of eliminating irrelevant variables plays a central role in several fields of investigation ranging from physics to chemistry, biology, astrophysics, and so on (for recent reviews see, e.g., ref. 1). This is a subject characterized by many facets and such subtle conceptual difficulties that for the sake of clarity it is convenient to focus on very simple systems. Let us consider the simple system described by

$$dx/dt = \varphi(x) + \psi(x)\xi \tag{1.1}$$

and let us make the usual assumption[2–14] that ξ is driven by

$$d\xi/dt = -\gamma\xi + f(t) \tag{1.2}$$

$f(t)$ is a white Gaussian noise described by

$$\langle f(0) f(t) \rangle = 2Q\delta(t) = 2\gamma \langle \xi^2 \rangle_{\text{eq}} \delta(t) \tag{1.3}$$

The main purpose of this paper is to discuss the physical condition under which it is possible to describe the stationary properties of the system of equations (1.1) and (1.2) in terms of an effective Fokker–Planck equation concerning the variable x alone.

After contraction over the variable ξ, the resulting equation of motion for the variable x turns out to be non-Markovian, thereby seemingly preventing us from using a Fokker–Planck equation, which is Markovian in nature. We shall show that, in spite of this, an effective Fokker–Planck description can be used to make correct predictions on the equilibrium distribution of x over a surprisingly extended range of values of the parameter

$$\tau \equiv 1/\gamma \tag{1.4}$$

which we shall refer to as the noise correlation time.

Our approach to this effective Fokker–Planck equation is of a perturbative nature, using the repartition of the complete Fokker–Planck operator $\mathscr{L}$ to separate the bidimensional system of equations (1.1) and (1.2) into a perturbation $\mathscr{L}_1$ and an unperturbed part $\mathscr{L}_0$.

For each perturbation order in $\mathscr{L}_1$ to be expressed through simple analytical equations, we must have recourse in general to a further expansion over the parameter τ. It is thus evident that the approach to the effective Fokker–Planck equation relies on a double perturbation expansion over $\mathscr{L}_1$ and τ. The resummation up to infinite order over τ while keeping the expansion over $\mathscr{L}$ at the second order is proven to coincide with the so-called best Fokker–Planck approximation (BFPA).[12–15] Note that the τ at the second order in $\mathscr{L}_1$ does not affect the Fokker–Planck structure.

On the contrary, the expansion over $\mathscr{L}_1$ at the fourth order results in a breakdown of the standard Fokker–Planck structure. To get rid of the influence of these corrections, Hanggi *et al.*[7] proposed a linearization assumption which has indeed the effect of making these higher order terms vanish.

In this paper we show that the global linearization of Hanggi *et al.*[7] is a poor approximation, which turns out to be totally unable to predict such an interesting effect as the transition from a one-mode to a two-mode distribution induced by the color of noise, which is an incontrovertible result of our analog and digital simulation techniques.

We show that a much more satisfactory way of getting rid of the terms of higher order in $\mathscr{L}_1$ is to make the assumption of local linearization (LL). This assumption has the twofold effect of making the terms of higher order in $\mathscr{L}_1$ vanish and of providing a straightforward resummation up to infinite order over τ. The resulting Fokker–Planck equation coincides with that recently proposed by Fox.[9] Our theory shows, however, that in spite of the pessimistic view of Fox, according to whom the use of this Fokker–Planck equation should be restricted to the short-τ region[9] it has a surprisingly extended regime of validity and for $\tau \to \infty$ the steady-state distribution turns out to coincide with the exact solution whereas the BFPA is correct only up to the first order in τ.

Our theoretical arguments are supported by the results of both analog and digital simulation.

The outline of this paper is as follows. Section 2 is devoted to illustrating the general aspects of our projection method. In Section 3 the problem of the resummation over τ at the second order in $\mathscr{L}_1$ is discussed. Section 4 is devoted to deriving the analytical expression of a crucial contribution at the fourth order in $\mathscr{L}_1$, which is comparable to the term of second order in τ of the BFPA. This is a central section, which leads us naturally to the LL approximation. In Section 5 we show that the LL approximation predicts exact results for $\tau \to \infty$. Section 6 supports our theoretical arguments by results of both analog and digital simulation (details on these techniques are provided in the Appendix). Concluding remarks on our results are found in Section 7.

2. THE PROJECTION METHOD

The first step of the projection method consists in writing the Fokker–Planck equation associated with Eq. (1.1). This reads

$$\frac{\partial}{\partial t}\rho(x, \xi; t) = (\mathscr{L}_a + \mathscr{L}_b + \mathscr{L}_1)\,\rho(x, \xi; t) \tag{2.1}$$

where

$$\mathscr{L}_a = -\frac{\partial}{\partial x}\varphi(x)$$

$$\mathscr{L}_b = \gamma\left(\frac{\partial}{\partial \xi}\xi + \langle \xi^2 \rangle_{\text{eq}} \frac{\partial^2}{\partial \xi^2}\right) \tag{2.2}$$

$$\mathscr{L}_1 = -\xi \frac{\partial}{\partial x}\psi(x)$$

and $\rho(x, \xi; t)$ is the probability distribution of the variables x and ξ at time t. The present approach is a perturbation theory with a perturbation term defined precisely by $\mathscr{L}_1$. Note that the explicit form of $\mathscr{L}_b$ of Eq. (2.2) is consistent with the Langevin equation of Eq. (1.2), implying both that $\langle \xi(0)\, \xi(t) \rangle$ is characterized by an exponential decay and ξ is a Gaussian noise.

The second step consists in writing Eq. (2.1) in the corresponding interaction picture ($\mathscr{L}_1$ is the natural interaction term). We thus have

$$\frac{\partial}{\partial t}\tilde{\rho}(t) = \mathscr{L}_1(t)\, \tilde{\rho}(t) \tag{2.3}$$

$$\tilde{\rho}(t) = \exp(-\mathscr{L}_0 t)\, \rho(t) \tag{2.4}$$

$$\mathscr{L}_1(t) = \exp(-\mathscr{L}_0 t)\, \mathscr{L}_1 \exp(\mathscr{L}_0 t) \tag{2.5}$$

where

$$\mathscr{L}_0 = \mathscr{L}_a + \mathscr{L}_b \tag{2.6}$$

The third step consists in applying to Eq. (2.3) the projection technique of Zwanzig[16] via the projection operator defined by $[\mathscr{L}_b \rho_{\text{eq}}(\xi) = 0]$

$$P\rho(x, \xi; t) = \rho_{\text{eq}}(\xi) \int d\xi\, \rho(x, \xi; t) \tag{2.7}$$

which leads to an equation of motion for

$$\sigma(x; t) = \int d\xi\, \rho(x, \xi; t) \tag{2.8}$$

with the following form:

$$\frac{\partial}{\partial t}\sigma(x; t) = \mathscr{L}_a \sigma(x; t) + \int_0^t K(t-s)\, \sigma(x; s)\, ds$$

$$= \mathscr{L}_a \sigma(x; t) + \int_0^t K(s)\, \sigma(x; t-s)\, ds \tag{2.9}$$

where

$$K(t-s) = \rho_{eq}^{-1}(\xi) \exp(-\mathscr{L}_a t)\, P\mathscr{L}_1(t) \times \overleftarrow{\exp}\left[\int_s^t dt'\,(1-P)\,\mathscr{L}_1(t')\right](1-P)\,\mathscr{L}_1(s)\,P\rho_{eq}(\xi)$$
$$= \rho_{eq}^{-1}(\xi)\, P\mathscr{L}_1 \exp(\mathscr{L}_0 t) \times \overleftarrow{\exp}\left[\int_s^t dt'\,(1-P)\,\mathscr{L}_1(t')\right](1-P)\exp(-\mathscr{L}_0 s)\,\mathscr{L}_1 P\rho_{eq}(\xi) \tag{2.10}$$

The arrows denote that we are dealing with a time-ordered exponential.[4] Note that we neglected the preparation term,[5] since we herein focus on the x-equilibrium distribution which is independent of that.

By expanding the ordered exponential in a Taylor power series and by employing the invariance of the multitime stationary correlation functions by time translation, one can write $K(s_0)$ as follows:

$$K(s_0) = \rho_{eq}^{-1}(\xi)\, P\mathscr{L}_1 \exp(\mathscr{L}_0 s_0)(1-P)\,\mathscr{L}_1 P\rho_{eq}(\xi) + \rho_{eq}^{-1}(\xi) \sum_{n=1}^{\infty} \int_0^{s_0} ds_1 \int_0^{s_1} ds_2 \int_0^{s_1} ds_n \times P\mathscr{L}_1 \exp(\mathscr{L}_0 s_0)(1-P)\,\mathscr{L}_1(s_1) \times (1-P)\,\mathscr{L}_1(s_{n-1})(1-P)\,\mathscr{L}_1(s_n)(1-P)\,\mathscr{L}_1 P\rho_{eq}(\xi) \tag{2.10'}$$

Upon adopting this expression for $K(s)$, Eq. (2.9) looks like that recently used by Sancho *et al.*[12]

To get rid of the time-convoluted nature of Eq. (2.9), we make the following approximation:

$$\int_0^t K(s)\,\sigma(x; t-s)\,ds \simeq \left\{\int_0^t ds\, K(s) \exp[-(\mathscr{L}_a + \mathbb{D})s]\right\} \sigma(x; t) \tag{2.11}$$

The operator $\mathbb{D}$ is introduced to take into account the fact that the history of $\sigma(x; t-s)$ (for $s>0$) not only depends on the unperturbed operator $\mathscr{L}_a$, but it is also dependent on the diffusion process geerated by the time-convoluted term itself. However, the operator $\mathbb{D}$ will exert its influence on the distribution $\sigma(x; t)$ only from the fourth order in $\mathscr{L}_1$ on. Note indeed that

[4] The time-ordered exponential is automatically involved when applying the Zwanzig approach to the interaction picture (see, e.g., ref. 17).

[5] The problem of the preparation term has been widely studied (see, e.g., ref. 18).

the lowest order contribution to both $K(s)$ and $\mathbb{D}$ is of the second order in $\mathscr{L}_1$. Thus, in the next section, devoted only to the second order in $\mathscr{L}_1$, we disregard $\mathbb{D}$ and we use the following approximation:

$$\int_0^t K(s)\,\sigma(x;t-s) \simeq \left[\int_0^t ds\; W(s)\right]\sigma(x;t) \tag{2.11'}$$

where

$$W(s) = K(s)\exp(-\mathscr{L}_a s) \tag{2.12}$$

At the second order in $\mathscr{L}_1$, $W(s)$ of Eq. (2.12) must rely on the following approximate expression for $K(s)$:

$$\begin{aligned} K(t-s) = {} & \rho_{\mathrm{eq}}^{-1}(\xi)\, P\mathscr{L}_1 \exp(\mathscr{L}_0 t)(1-P) \\ & \times \exp(-\mathscr{L}_0 s)\, \mathscr{L}_1 P \rho_{\mathrm{eq}}(\xi) \end{aligned} \tag{2.13}$$

In conclusion, at the second order in $\mathscr{L}_1$, we have

$$\frac{\partial}{\partial t}\sigma(x;t) = \left[-\frac{\partial}{\partial x}\varphi(x) + \left(\int_0^t W(s)\,ds\right)\right]\sigma(x;t) \tag{2.14}$$

with $W(s)$ defined by Eqs. (2.12) and (2.13).

Equation (2.14) with $W(s)$ defined by Eqs. (2.12) and (2.13) is easily recognized to coincide with the second-order cumulant expansion recommended by Mukamel *et al.*[19] to study Gaussian processes. It must also be stressed that Eq. (2.14) with $W(s)$ defined by Eqs. (2.12) and (2.13), when used to determine the time evolution of $\langle x^n(t)\rangle$, appears to be clearly related to the equation of motion proposed by van Kampen.[20] This can be made clear by making explicit Eq. (2.14) in the case where $\mathscr{L}_a$ and $\mathscr{L}_1$ are given the form of Eq. (2.2). In this case we obtain that Eq. (2.14), supplemented by Eqs. (2.12) and (2.13), reads

$$\begin{aligned} \frac{\partial}{\partial t}\sigma(x;t) = {} & \mathscr{L}_a \sigma(x;t) \\ & + \left[\int_0^t ds\, \langle \xi(0)\,\xi(s)\rangle \frac{\partial}{\partial x}\psi \exp(\mathscr{L}_a s)\frac{\partial}{\partial x}\psi \exp(\mathscr{L}_a s)\right]\sigma(x;t) \end{aligned} \tag{2.15}$$

where $\langle\cdots\rangle$ means average over the equilibrium distribution $\rho_{\mathrm{eq}}(\xi)$ and $\xi(t) = \exp(\mathscr{L}_b^{+} t)$ [note that in the stationary case, $\langle \xi(s)\,\xi(0)\rangle = \langle \xi(t)\,\xi(t-s)\rangle$, which make complete the equivalence with Eq. (10.4) of ref. 20 once the upper limit of time integration t is replaced by infinity].

The problem under investigation in this paper is characterized by the two perturbation parameter $D = \langle \xi^2 \rangle \tau$ (intensity of diffusion) and τ (noise correlation time). It is generally thought (see, for instance, the general discussion of ref. 20) that both parameters must be kept very small for Eq. (2.14) [supplemented by Eqs. (2.12) and (2.13)] to work. We shall show, on the contrary, that if D is kept small, a condition of local linearization holds which makes it possible to explore successfully even the long-τ region (see Section 4).

3. AN EXPANSION OVER ALL ORDERS IN τ

In this section we show how to carry out the resummation over all orders in τ at the second order in $\mathscr{L}_1$.

We introduce an exponential $\exp[\Pi(x)s]$, which from a purely formal point of view allows us to write

$$\frac{\partial}{\partial x}\psi(x)\exp(\mathscr{L}_a s)\frac{\partial}{\partial x}\psi(x)\exp(-\mathscr{L}_a s) = \frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\exp[\Pi(x)s] \tag{3.1}$$

Let us consider the superoperator $\mathscr{L}_a^x$ defined by

$$\mathscr{L}_a^x A = \mathscr{L}_a A - A\mathscr{L}_a \tag{3.2}$$

where A denotes a generic differential operator. We can formally write

$$\begin{aligned}\exp(\mathscr{L}_a^x t)A &= (1 + \mathscr{L}_a^x t + \tfrac{1}{2}(\mathscr{L}_a^x)^2 t^2 + \cdots)A \\ &= A(1 + \Pi^{(1)}t + \tfrac{1}{2}\Pi^{(2)}t^2 + \cdots)\end{aligned} \tag{3.3}$$

which can be written as

$$\exp(\mathscr{L}_a^x t)A = A\exp(\Pi t) \tag{3.4}$$

thereby providing

$$\begin{aligned}\exp(\mathscr{L}_a t)A &= A\exp(\Pi t)\exp(\mathscr{L}_a t) \\ A\exp(-\mathscr{L}_a t) &= \exp(-\mathscr{L}_a t)A\exp(\Pi t)\end{aligned} \tag{3.5}$$

Equation (3.1) is obtained precisely from Eq. (3.4) with $A = (\partial/\partial x)\psi(x)$.

From Eq. (3.3) we see that the exponential $\exp(\Pi t)$ is defined through its expansion in a Taylor series

$$\exp(\Pi t) = 1 + \Pi^{(1)}t + \tfrac{1}{2}\Pi^{(2)}t^2 + \cdots \tag{3.6}$$

with

$$\Pi^{(0)} = 1 \tag{3.6'}$$

$$\Pi^{(i)} = (1/A)(\mathscr{L}_a^x)^i A, \qquad i > 1 \tag{3.6''}$$

In the case under study, $A = (\partial/\partial x)\,\psi(x)$, we find

$$\begin{aligned} &\Pi^{(i)}(x) = \pi^{(i)}(x)/\psi(x), \qquad i = 0, 1, 2 \\ &\pi^{(0)}(x) = \psi(x) \\ &\pi^{(i)}(x) = \varphi'(x)\,\pi^{(i-1)}(x) - \pi^{(i-1)'}(x)\,\varphi(x), \qquad i > 0 \end{aligned} \tag{3.7}$$

The property of Eq. (3.1) can be used to rewrite Eq. (2.15) (with the upper limit of integration t replaced by infinity) as follows:

$$\frac{\partial}{\partial t}\sigma(x;t) = \left\{ -\frac{\partial}{\partial x}\varphi(x) + D\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)[1 - \tau\Pi(x)]^{-1} \right\}\sigma(x;t) \tag{3.8}$$

or in the equivalent form

$$\frac{\partial}{\partial t}\sigma(x;t) = \left[-\frac{\partial}{\partial x}\varphi(x) + D\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\Phi(x) \right]\sigma(x;t) \tag{3.9}$$

with

$$\Phi(x) = \psi(x)\sum_{n=0}^{\infty}\tau^n\Pi^{(n)}(x) \tag{3.10}$$

Equation (3.9), supplemented by Eqs. (3.10) and (3.7), is known as the best Fokker–Planck approximation (BFPA).[12–15]

Resummation up to infinity over τ means expressing the series of Eq. (2.10) via an analytical expression. For instance, in the special case of the periodic potential

$$\varphi(x) = k\cos(\omega_0 x) \tag{3.11}$$

with $\psi(x) = 1$, it is straightforward to show that

$$\Phi(x) = [1 - k\omega_0[\sin(\omega_0 x)]/\gamma][1 - (k\omega_0/\gamma)^2]^{-1} \tag{3.12}$$

Peacok-Lopez *et al.*[20] succeeded in providing an analytical expression for $\Phi(x)$ even in the case where $\varphi(x) = \alpha x - \beta x^3$ $(\alpha > 0)$.

We show, however, in the next section that the exact expression of $\Phi(x)$, when used to derive the equilibrium distribution, provides incorrect

information on the contributions to it at orders higher than the first in τ. Thus, it will turn out to be convenient to use what seemingly is an approximation to the exact expression of $\Phi(x)$.[14,15]

4. THE CONTRIBUTION TO DIFFUSION AT THE FOURTH ORDER IN $\mathscr{L}_1$

From a purely formal point of view Eq. (2.9) can be rewritten as follows [see Eqs. (2.11) and (2.12)]:

$$\frac{\partial}{\partial t}\sigma(x;t)=\left\{-\frac{\partial}{\partial x}\varphi(x)+\int_0^t W(s)\exp[-\mathbb{Q}(x)s]\,ds\right\}\sigma(x;t) \tag{4.1}$$

where [we recall that $\mathscr{L}_a=-(\partial/\partial x)\,\varphi(x)$]

$$\exp[-\mathbb{Q}(x)s]\equiv\exp(\mathscr{L}_a s)\exp[-(\mathscr{L}_a+\mathbb{D})s] \tag{4.2}$$

and $\mathbb{D}$ is precisely the diffusion operator we want to built up. Its implicit expression is

$$\mathbb{D}\equiv\int_0^\infty W(s)\exp[-\mathbb{Q}(x)s]\,ds \tag{4.3}$$

We aim at making a calculation up to the fourth order in the interaction $\mathscr{L}_1$ of Eq. (2.2). Thus we write

$$\mathbb{D}=\int_0^\infty W_2(s)\exp[-\mathbb{Q}^2(x)s]\,ds+\int_0^\infty W_4(s)\,ds \tag{4.4}$$

where W_2 and W_4 are the second- and fourth-order contributions to the operator W defined by Eqs. (2.12) and (2.10′), and $\mathbb{Q}_2(s)$ is defined by

$$\exp[-\mathbb{Q}_2(x)s]\equiv\exp(\mathscr{L}_a s)\exp[-(\mathscr{L}_a+\mathbb{D}_2)s] \tag{4.5}$$

This definition of $\mathbb{Q}_2$ now makes sense, because $\mathbb{D}_2$ is defined by

$$\mathbb{D}_2=\int_0^\infty W_2(s)\,ds=D\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\Phi(x) \tag{4.6}$$

As to the second term on the rhs of Eq. (4.4), we note that from the fourth-order contribution to $K(s_0)$ of Eq. (2.10′) and by using Eq. (2.12) we obtain

$$W_4(s_0)=\int_0^{s_0}ds_1\int_0^{s_1}ds_2\,X(s_0,s_1,s_2)\,Y(s_0,s_1,s_2) \tag{4.7}$$

where

$$X(s_0, s_1, s_2) = \langle \xi e^{\mathscr{L}_b(s_0 - s_1)} \xi e^{\mathscr{L}_b(s_1 - s_2)} \xi e^{\mathscr{L}_b s_2} \xi \rangle - \langle \xi e^{\mathscr{L}_b s_0} \xi e^{\mathscr{L}_b(s_1 - s_2)} \rangle \langle \xi e^{\mathscr{L}_b s_2} \rangle \tag{4.8}$$

$$Y(s_0, s_1, s_2) = \frac{\partial}{\partial x} \psi(x)\, e^{\mathscr{L}_a(s_0 - s_1)} \frac{\partial}{\partial x} \psi(x)\, e^{\mathscr{L}_a(s_1 - s_2)} \times \frac{\partial}{\partial x} \psi(x)\, e^{\mathscr{L}_a s_2} \frac{\partial}{\partial x} \psi(x)\, e^{-\mathscr{L}_a s_0} \tag{4.9}$$

The calculation of X can be done with the standard rules, consisting of applying $\mathscr{L}_b$ on the left.[22] The final result is

$$X(s_0, s_1, s_2) = 2\langle \xi^2 \rangle^2 e^{-\gamma(s_0 - s_1)} e^{-2\gamma(s_1 - s_2)} e^{-\gamma s_2} \tag{4.10}$$

More subtle difficulties are involved with the calculation of $Y(s_0, s_1, s_2)$. First, we apply the commutation rule of Eq. (3.5). This allows us to write Eq. (4.9) as follows:

$$Y(s_0, s_1, s_2) = \frac{\partial}{\partial x} \psi(x) \frac{\partial}{\partial x} \psi(x)\, e^{\Pi(s_0 - s_1)} \times \frac{\partial}{\partial x} \psi(x)\, e^{\Pi(s_0 - s_1)} \frac{\partial}{\partial x} \psi(x)\, e^{\Pi s_0} \tag{4.11}$$

To be in a position to do the integration over the times s_0, s_1, and s_2 associated with Eqs. (4.7) and (4.4) (second term on the rhs of this latter equation), it is convenient to put together the exponentials $\exp(\Pi s)$ appearing in Eq. (4.11). We must thus solve the problem of commuting $\exp[\Pi(x)s]$ with $\partial/\partial x$. To this purpose, we are naturally led to define a hierarchy of "functions" $\Pi(n)$ [$\Pi(0)$ has to be identified with Π defined by Eqs. (2.6) and (2.17)] by remarking that

$$\begin{aligned} \frac{\partial}{\partial x} e^{\Pi(n)s} &= \frac{\partial}{\partial x}\left[1 + \Pi^{(1)}(n)s + \frac{1}{2}\Pi^{(2)}(n)\, s^2 + \cdots\right] \\ &= \left[1 + \Pi^{(1)}(n)s + \frac{1}{2}\Pi^{(2)}(n)\, s^2 + \cdots\right]\frac{\partial}{\partial x} \\ &\quad + \left[\Pi^{(1)'}(n)s + \frac{1}{2}\Pi^{(2)'}(n)\, s^2 + \cdots\right] \\ &= e^{\Pi(n)s}\frac{\partial}{\partial x} + s\Pi^{(1)'}(n)\{1 + [\Pi^{(2)'}(n)]/[2\Pi^{(1)'}(n)] + \cdots\} \end{aligned} \tag{4.12}$$

This shows that the we can use the commutation rules

$$\frac{\partial}{\partial x} e^{\Pi(n)s} = s\Pi^{(1)'}(n)\, e^{\Pi(n+1)s} + e^{\Pi(n)s} \frac{\partial}{\partial x} \tag{4.13}$$

$$e^{\Pi(n)s} \frac{\partial}{\partial x} = e^{\Pi(n)s} - s\Pi^{(1)'}(n)\, e^{\Pi(n+1)s} \tag{4.14}$$

provided that the term $n+1$ of the hierarchy is defined in terms of the preceding one as follows:

$$\Pi^{(m)'}(n+1) = [\Pi^{(m+1)'}(n)]/\{(m+1)[\Pi^{(m)'}(n)]\}, \qquad m, n \geqslant 0 \tag{4.15}$$

By using Eqs. (4.13) and (4.14), we put the last and the first exponential $\exp(\Pi s)$ of Eq. (4.11) in the same central position as $\exp[\Pi(s_0 - s_2)]$, at the price of producing third- and second-order derivatives (in addition to the fourth-order derivatives which naturally appear at the fourth order in $\mathscr{L}_1$),

$$\begin{aligned} Y(s_0, s_1, s_2) = {} & \frac{\partial}{\partial x}\psi(x) \frac{\partial}{\partial x}\psi(x) \frac{\partial}{\partial x}\psi(x)\, e^{\Pi(3s_0 - s_1 - s_2)} \frac{\partial}{\partial x}\psi(x) \\ & + \frac{\partial}{\partial x}\psi(x) \frac{\partial}{\partial x}\psi(x) \frac{\partial}{\partial x}\psi(x)\, e^{\Pi(2s_0 - s_1 - s_2)} \psi(x)\, \Pi^{(1)'} s_0 e^{\Pi(1)s_0} \\ & - \frac{\partial}{\partial x}\psi(x) \frac{\partial}{\partial x}\psi^2(x)\, \Pi^{(1)'}(x)(s_0 - s_1) \\ & \times e^{\Pi(1)(s_0 - s_1) + \Pi(2s_0 - s_2)} \frac{\partial}{\partial x}\psi(x) \\ & - \frac{\partial}{\partial x}\psi(x) \frac{\partial}{\partial x}\psi^2(x)\, \Pi^{(1)'}(x)(s_0 - s_1) \\ & \times e^{\Pi(1)(s_0 - s_1) + \Pi(s_0 - s_2) + \Pi(1)s_0}\, \Pi^{(1)'} s_0 \end{aligned} \tag{4.16}$$

The four terms on the rhs of Eq. (4.16) have to be multiplied by X of Eq. (4.10) and then the integration over s_0, s_1, and s_2 necessary to determine

$$\int_0^\infty W_4(s_0)\, ds_0$$

must be done, generating four terms which we shall refer to as α_1, α_2, α_3, and α_4, respectively. We now remark that α_1 is a term with a fourth-order derivative proportional to $D^2(a_1\tau + a_2\tau^2 + a_3\tau^3 + \cdots)$. From the first term

on the rhs of Eq. (4) we draw terms which will be proven to cancel with α_1 up to the order $D^2\tau^2$.

From our demonstration it will appear plausible that the cancellation holds up to infinite order in τ. However, we are interested only in recovering from the rearrangement of

$$\int_0^\infty W_2(s) \exp[-\mathbb{Q}(x)s]\, ds$$

the first term on the rhs of Eq. (4.4), third-order derivatives proportional to $D^2\tau^2$ (third-order derivatives proportional to $D^2\tau^n$ with $n>2$ would be replaced with the renormalization arguments used later in this section by standard diffusion terms proportional to $D\tau^n$, i.e., of order higher than that to which we limit ourselves in this paper, $D\tau^2$). The terms α_2 and α_3 are precisely third-order derivatives proportional to $D^2\tau^2$ and we must focus our attention on them. The third term, α_4, on the contrary, is a second-order derivative proportional to $D^2\tau$. This term can be neglected simply by keeping the noise intensity very weak, in that the standard term stemming from the calculation of the order $\mathscr{L}_1^2$ is proportional to D.

Let us focus our attention on the first term on the rhs of Eq. (4.4). Let us make the assumption of neglecting terms of order higher than D. The terms of order higher than D must cancel with contributions stemming from the perturbation calculation at the order $\mathscr{L}_1^6$, $\mathscr{L}_1^8$, and so on (in this paper we limit ourselves to a calculation up to the order $\mathscr{L}_1^4$).

By developing the exponential $\exp(-\mathbb{Q}_2 s)$ as defined by Eq. (4.5), we obtain (under the assumption above of neglecting contributions of the order D^n with $n \geqslant 2$)

$$\begin{aligned}
&\int_0^\infty W_2(s) \exp[-\mathbb{Q}_2(x)s]\, ds \\
&\quad = \hat{W}_2(0) + \left[\frac{d}{dz}\hat{W}_2(z)\right]_{z=0} \hat{W}_2(0) \\
&\qquad + \frac{1}{2}\left[\frac{d^2}{dz^2}\hat{W}_2(z)\right]_{z=0} [\hat{W}_2(0), \mathscr{L}_a] \\
&\quad = D\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\frac{1}{1-\tau\Pi} + D\tau\left\{-\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\frac{1}{(1-\tau\Pi)^2}\right. \\
&\qquad \times\left[\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\frac{1}{1-\tau\Pi}\right] - \frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\tau\frac{1}{(1-\tau\Pi)^3} \\
&\qquad \left.\times\left[\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\frac{1}{1-\tau\Pi}\frac{\partial}{\partial x}\psi(x)\right]\right\} \qquad (4.17)
\end{aligned}$$

[The symbol $\hat{W}_2(z)$ denote the Laplace transform of $W_2(t)$ at the frequency z.] The first term on the rhs of the last equality is the standard contribution of the BFPA. The second term on the rhs of the last equality, $D\tau\{\cdots\}$, cancels exactly (up to the second order in τ) with α_1. This means that α_2 and α_3 are the only relevant corrections to the BFPA.

We thus obtain

$$\frac{\partial}{\partial t}\sigma(x;t)=\left[-\frac{\partial}{\partial x}\varphi(x)+D\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\frac{1}{1-\tau\Pi}\right.$$
$$\left.+\frac{3}{2}D^2\tau^2\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)^2(x)\,\Pi^{(1)'}\right]\sigma(x;t) \tag{4.18}$$

The last term on the rhs of Eq. (4.18) derives from $\alpha_2+\alpha_3$.

From Eq. (4.18) we immediately see that a natural way of getting rid of the nonstandard diffusion term on the rhs is to assume

$$\Pi^{(1)'}(x)\simeq 0 \tag{4.19}$$

This does not mean that $\Pi^{(1)}(x)$ is constant throughout the whole range: $-\infty\leqslant x\leqslant+\infty$. We only need that D be weak enough to make the diffusion around a certain position x very slow and to oblige the Brownian particle to "feel" different values of $\Pi^{(1)}$ virtually constant for an extended period of time. Of course, particles in different regions are allowed to "feel" different values of $\Pi^{(1)}(x)$. By analogy with the additive case, where [due to the definition of Eq. (3.7)] $\Pi^{(1)}(x)=\varphi'$, we call this approximation local linearization (LL). This approximation eliminates the terms of order higher than the second in $\mathscr{L}_1$, producing the same effect as the global linearizations

$$\Pi^{(1)}(x)\simeq\langle\Pi^{(1)}(x)\rangle \tag{4.20}$$

$$\Pi^{(1)}(x)\simeq\Pi^{(1)}(x_s) \tag{4.21}$$

[where $x_s=\langle x(t)\rangle$] suggested by refs. 7 and 23, respectively.

On the other hand, if Eq. (4.19) holds, from Eqs. (3.7) we immediately see that $\Phi(x)$ of Eq. (4.10) reads

$$\Phi(x)=\psi(x)/[1-\tau\Pi^{(1)}(x)] \tag{4.22}$$

which coincides precisely with the Fox theory.[9]

If the LL approximation does not hold and the fourth-order correction term of Eq. (4.18) must be taken into account, this corrects the BFPA

at the second order in τ. To show this, let us consider the white noise approximation to the BFPA,

$$\frac{\partial}{\partial t}\sigma(x;t)=\left[-\frac{\partial}{\partial x}\varphi(x)+D\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\right]\sigma(x;t) \tag{4.23}$$

It is immediately seen that the corresponding equilibrium distribution $\sigma_{eq}(x)$ satisfies the relation

$$\sigma'_{eq}(x)=\frac{\varphi(x)-D\psi(x)\,\psi'(x)}{D\psi^2(x)}\sigma_{eq}(x) \tag{4.24}$$

Let us assume that τ is so short as to make the equilibrium distribution of Eq. (4.18) very close to the white noise approximation of Eq. (4.24). Neglecting terms with second-order derivatives proportional to D^2 and using Eq. (4.24), we obtain that the last term on the rhs of Eq. (4.18) can be transformed by the following series of obvious approximations:

$$\begin{aligned}
&\frac{3}{2}D^2\tau^2\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi^2(x)\,\Pi^{(1)'}(x)\,\sigma(x;t)\\
&\quad\simeq\frac{3}{2}D^2\tau^2\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi^3(x)\,\Pi^{(1)'}(x)\frac{\partial}{\partial x}\sigma(x;t)\\
&\quad\simeq\frac{3}{2}D^2\tau^2\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi^3(x)\,\Pi^{(1)'}(x)\frac{\partial}{\partial x}\sigma_{eq}(x;t)\\
&\quad\simeq\frac{3}{2}D^2\tau^2\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi^3(x)\,\Pi^{(1)'}(x)\frac{\varphi(x)-D\psi(x)\,\psi'(x)}{D\psi^2(x)}\sigma_{eq}(x;t)\\
&\quad=\frac{3}{2}D^2\tau^2\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi^3(x)\,\Pi^{(1)'}(x)\frac{\varphi(x)-D\psi(x)\,\psi'(x)}{D\psi^2(x)}\sigma(x;t)\\
&\quad=\frac{3}{2}D\tau^2\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\,\varphi(x)\,\Pi^{(1)'}(x)\,\sigma(x;t)
\end{aligned} \tag{4.25}$$

We thus obtain a contribution comparable to the second order in τ provided by the BFPA. This is incontrovertible evidence that the equilibrium distribution provided by the BFPA is correct only up to the first order in τ. In the next section we show that the LL approximation, on the contrary, for $\tau\to\infty$ leads to the exact steady-state distribution.

5. STUDY OF THE EQUILIBRIUM DISTRIBUTION IN THE INTERVAL $-\infty\leqslant x\leqslant+\infty$

From now on we focus our attention on the additive case $\psi(x)=1$ with $\varphi(x)=\alpha x-\beta x^3$ $(\alpha>0)$, which corresponds to a quartic potential.

Note that the general equilibrium solution of Eq. (3.9) in the additive case is

$$\sigma_{eq}(x) = \sigma_0 \exp\left[\int_0^t \frac{\varphi(x')}{D\Phi(x)}\,dx' - \log \Phi(x)\right] \tag{5.1}$$

To study the properties of this equilibrium distribution, we adopt the expression for $\Phi(x)$ stemming from the LL approximation,

$$\Phi(x) = \frac{1}{1-\tau\varphi'(x)} \tag{5.2}$$

We make this choice for the following reasons:

1. Equation (5.2) coincides up to the first order in τ with the expression for $\Phi(x)$ resulting from the BFPA.

2. As already stressed by Fox,[9] Eq. (5.2) leads to uniform convergence.

3. The contributions of nth order in τ with $n>1$ stemming from the BFPA are not reliable (in the preceding section we showed that from the fourth order in $\mathscr{L}_1$ we draw a contribution comparable to that of second order in τ of the BFPA).

4. For $\tau \to \infty$, Eq. (5.1) with $\Phi(x)$ provided by Eq. (5.2) (i.e., the result of the LL approximation) leads to exact predictions.

Properties 1–3 have already been shown. We must now prove property 4.

When τ is much larger than the relaxation time of x, the system is certainly given enough time to reach its deterministic equilibrium position at a certain value of ξ,

$$\alpha x - \beta x^3 = -\xi \tag{5.3}$$

before ξ takes on new values. The probability for the Gaussian variable ξ to be found between the values ξ and $\xi + d\xi$ is

$$d\rho_{eq}(\xi) = (2\tau\langle\xi^2\rangle_{eq})^{-1/2} \exp(-\xi^2/2\langle\xi^2\rangle_{eq})\,d\xi \tag{5.4}$$

Note that from Eq. (5.3) we obtain

$$d\xi = (3\beta x^2 - \alpha)\,dx \tag{5.5}$$

which leads us to determine the probability for the variable x to be found between x and $x+dx$ via the expression

$$d\sigma_{eq}(x) \propto (3\beta x^2 - \alpha) \exp[-(\beta x^3 - \alpha x)^2/2\langle\xi^2\rangle_{eq}]\,dx \tag{5.6}$$

In other words, the equilibrium distribution of the variable x is

$$\sigma_{eq}(x) \propto (3\beta x^2 - \alpha) \exp[-(\beta x^3 - \alpha x)^2 / 2\langle \xi^2 \rangle_{eq}] \tag{5.7}$$

When $3\beta x^2 - \alpha < 0$, we must assume $\sigma_{eq}(x) = 0$. This can be justified by remarking that the solution of Eq. (5.3) is characterized by three roots, corresponding to two stable positions and an unstable one. With increasing ξ ($\xi < 0$) the unstable equilibrium position and the stable one on the positive side of x merge into one another at

$$x = (\alpha/3\beta)^{1/2} \tag{5.8}$$

while the stable position on the negative side reaches the value $x = -2(\alpha/3\beta)^{1/2}$; the reverse effect takes places when $\xi > 0$. Stable and unstable equilibrium positions merge into one another at

$$x = -(\alpha/3\beta)^{1/2} \tag{5.9}$$

whereas the equilibrium position on the positive side of the x axis reaches the limit value $x = 2(\alpha/3\beta)^{1/2}$. This shows that no stable equilibrium position is allowed between $x = -(\alpha/3\beta)^{1/2}$ and $x = (\alpha/3\beta)^{1/2}$. On the hand, the transient time in this central region depends on the parameters α and β and is finite. For $\tau \to \infty$ this transient time turns out to be negligible compared to the residence time of the particle in the side region. Thus we are led to

$$\begin{aligned} &\sigma_{eq}(x) \propto (3\beta x^2 - \alpha) \exp[-(\beta x^3 - \alpha x)^2 / 2\langle \xi^2 \rangle_{eq}], \qquad 3\beta x^2 > \alpha \\ &\sigma_{eq}(x) = 0 \qquad \text{for} \quad 3\beta x^2 < \alpha \end{aligned} \tag{5.10}$$

Note that Eq. (5.1) with $\Phi(x)$ given by Eq. (5.2) for $\tau \to \infty$ leads precisely to the same equilibrium distribution as Eq. (5.10) (provided that we use the same convention when $3\beta x^2 < \alpha$). This is equivalent to proving property 4 to be true.

To discuss the physical properties of the equilibrium distribution predicted by the LL approximation, we examine separately the cases $\alpha > 0$ and $\alpha = 0$.

(i) $\alpha > 0$. In the white noise limit the equilibrium distribution is characterized by two peaks symmetrically placed around $x = 0$ at $x = \pm(\alpha/\beta)^{1/2}$. Replacing Eq. (5.2) into Eq. (5.1) (i.e., using the LL approximation) and assuming the condition

$$D \ll \alpha\gamma/3\beta \tag{5.11}$$

to be satisfied, we obtain that the distance between the two peaks increases upon increase of D. The peaks at $x>0$ and $x<0$ shift, respectively, toward larger and smaller values of x by the amount

$$\Delta x_{\max} = 3(\beta/\alpha)^{1/2} D/[\gamma(1+2\alpha/\gamma)^2] \tag{5.12}$$

The maximum possible shift (compatible with the crucial condition $\tau < 1/\alpha$) is then obtained when $\alpha/\gamma = 1/2$.

From Eq. (5.12) we also obtain that for $\tau \to 0$

$$\Delta x_{\max} = 3(\beta/\alpha)^{1/2} D/\gamma \tag{5.13}$$

which shows that no shift is exhibited in the white noise limit.

For $\tau \to \infty$, on the contrary, we see from Eq. (5.10) that the equilibrium distribution is distinctly bimodal and the two peaks are shifted by the quantity

$$\Delta x_{\max} = \tfrac{3}{4}(\beta/\alpha)^{1/2} \langle \xi^2 \rangle_{\text{eq}}/\alpha^2 \tag{5.14}$$

By using Eq. (5.14) and (5.12) we reach the conclusion that upon increase of τ with D kept constant, $\Delta x_{\max}$ increases and after reaching a maximum value decreases so as to vanish again for $\tau \to \infty$. Note indeed that if D is kept constant, $\langle \xi^2 \rangle_{\text{eq}}$ becomes vanishingly small in the limiting case $\tau = 0$. If, on the contrary, $\langle \xi^2 \rangle_{\text{eq}}$ is kept constant, the shift $\Delta x_{\max}$ attains a non-vanishing limit for $\tau \to \infty$.

(ii) $\alpha = 0$. In this case the white noise limit is characterized by a monomodal distribution. At finite values of τ the equilibrium distribution is actually bimodal and this character becomes increasingly evident upon increase of τ. This phenomenon has been noted by Lugiato and Horowitz[24] by using the version of the BFPA corresponding to the theory of the Barcelona group.[11] The result of Lugiato and Horowitz is therefore confined to the short-τ region, whereas the LL approximation allows us to explore the whole range $0 \leqslant \tau \leqslant \infty$.

Let us denote by $\pm x_{\max}$ the positions of the two peaks and call ρ_0 the value of $\sigma_{\text{eq}}(x)$ at $x=0$; let us denote by Δ the depth of the dip between the two peaks (Fig. 1). We then obtain that when the condition

$$\beta D/\gamma^2 \ll 1 \tag{5.15}$$

is satisfied, $x_{\max}$ and Δ/ρ_0 are given by (using the LL approximation)

$$x_{\max} \sim (6D/\gamma)^{1/2} \tag{5.16}$$

$$\Delta/\rho_0 \sim 9\beta D/\gamma^2 \tag{5.17}$$

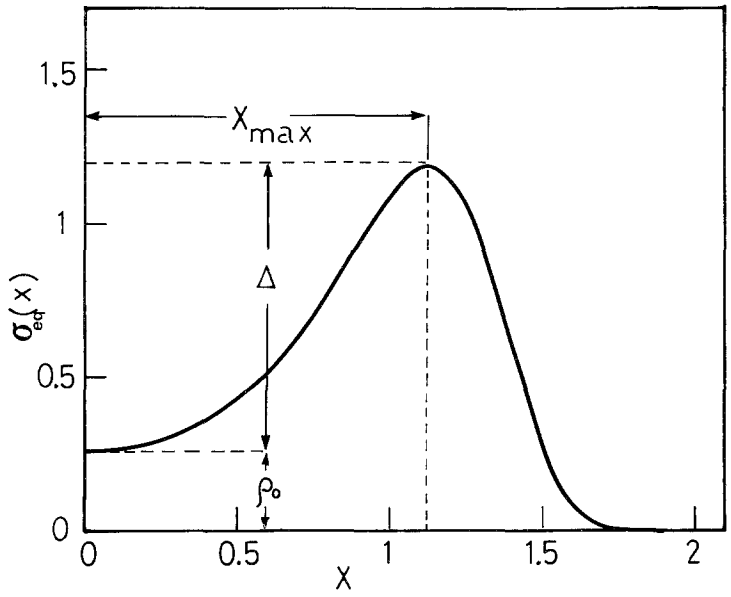


Fig. 1. Schematic picture of the bimodal equilibrium distribution.

This shows that for $\tau \to 0$, Δ/ρ_0 becomes negligible, making the equilibrium distribution indistinguishable from a monomodal distribution.

In the limiting case $\tau \to \infty$ we obtain from the equilibrium distribution of Eq. (5.10) that the peaks generated by the color of the noise reach the limit

$$x_{\max} = (\tfrac{2}{3}\langle \xi^2 \rangle_{eq}/\beta^2)^{1/2} \tag{5.18}$$

By using Eqs. (5.16) and (5.18) we can see that if D is kept constant, than $x_{\max}$ (like $\Delta x_{\max}$ of the case $\alpha \neq 0$) must exhibit a nonmonotonic behavior as a function of τ. From Eq. (5.16) we see that at $\tau = 0$, $x_{\max}$ vanishes. Upon increase of τ, $x_{\max}$ increases. Then it must reach a maximum value and decrease so as to vanish again for $\tau \to \infty$, as indicated by Eq. (5.18). If, on the contrary, we keep $\langle \xi^2 \rangle_{eq}$ constant, $x_{\max}$ reaches a nonvanishing limit for $\tau \to \infty$.

It is straightforward to show that the BFPA does not attain the exact prediction of Eq. (5.10) for $\tau \to \infty$. Since the BFPA and the LL approximation lead to equilibrium distributions coinciding at the first order in τ, their predictions must be virtually indistinguishable from one another in the short-τ region. At moderately large values of τ their predic-

tions start departing from one another, and from there on the LL approximation is expected to provide results systematically more accurate than the BFPA.

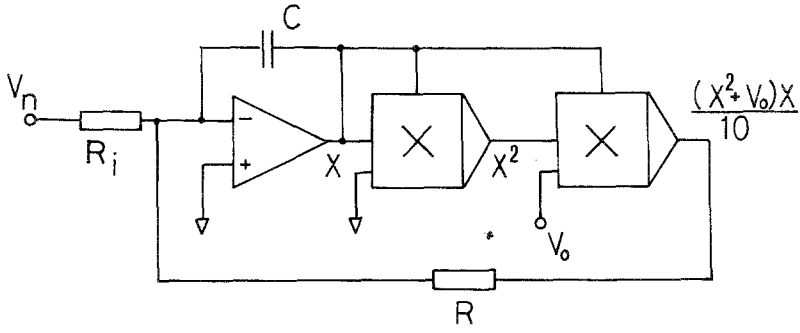
6. COMPARISON BETWEEN THEORY AND THE RESULTS OF ANALOG AND DIGITAL SIMULATIONS

This section is devoted to showing that the results of digital and analog simulation (details on these techniques are given in the Appendix) support the theoretical results of the preceding sections.

Figure 2 illustrates that the theoretical predictions on the change of the peaks of the probability distribution upon increase of τ are correct. The global linearization (7) cannot account for this interesting effect.

The transition from the one- to the two-mode distribution predicted by the LL approximation is confirmed by the results of Fig. 3. We see that the agreement is remarkably good even in the case of extremely colored noise ($\tau = 22$). In the short-τ region the BFPA is indistinguishable from the LL approximation, whereas the global linearization (7) is grossly wrong.

Of special interest are the results of Fig. 4 concerning the cosine case of Eq. (3.11). In this case the exact resummation of the BFPA is known [Eq. (3.12)]. We see from Figs. 4 that in accordance with our theoretical predictions the LL approximation turns out to be more accurate than the BFPA.



$$\dot{X} = \frac{V_o X}{10RC} - \frac{X^3}{10RC} - \frac{V_n}{R_i C}$$

Fig. 2. The shifts of the peaks of the bimodal equilibrium distribution as a function of the correlation time τ. (—) The result of the global linearization.[7] (- -) The prediction of the LL approximation. (○) The results of digital simulation; (▲) the results of analog simulation. $D = 0.5$, $\alpha = 1$, $\beta = 1$.

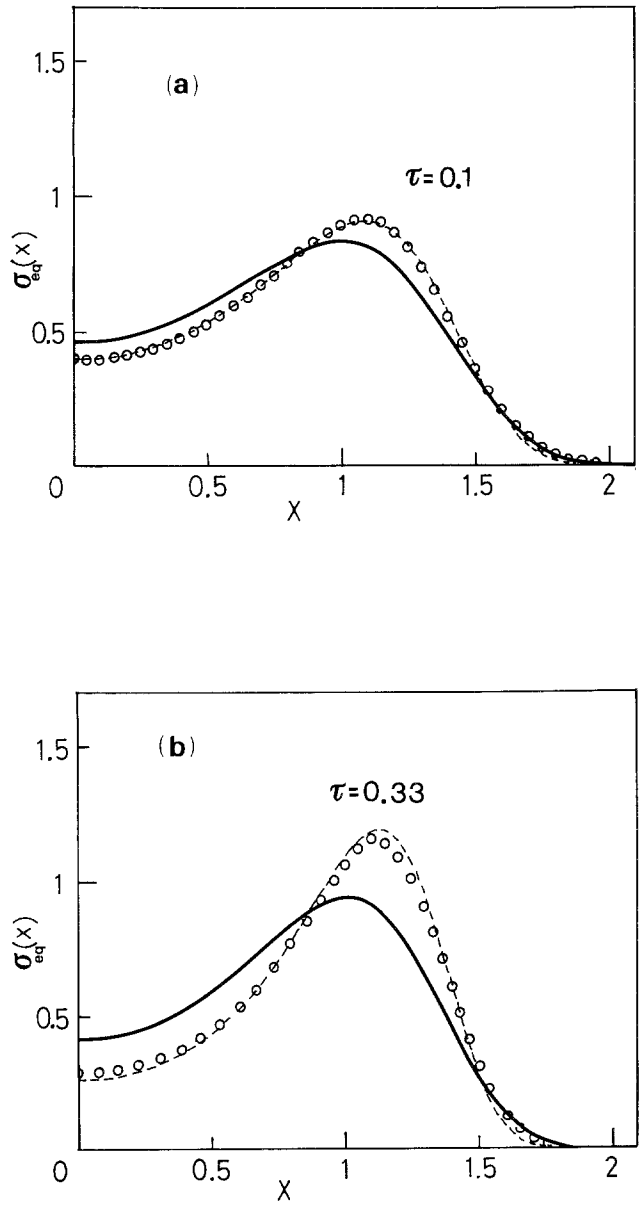


Fig. 3. The transformation of a monomodal equilibrium distribution into a bimodal one as an effect of colored noise. (—) The prediction of the global linearization.[7] (- -) The prediction of the LL approximation.. (○) The results of digital simulation; (▲) the results of analog simulation. $D = 0.5$, $\alpha = 0$, $\beta = 1$.

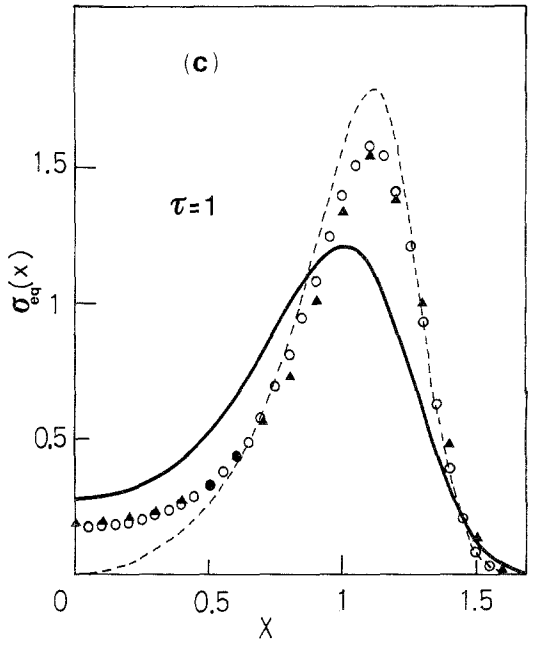


Fig. 3 (*continued*)

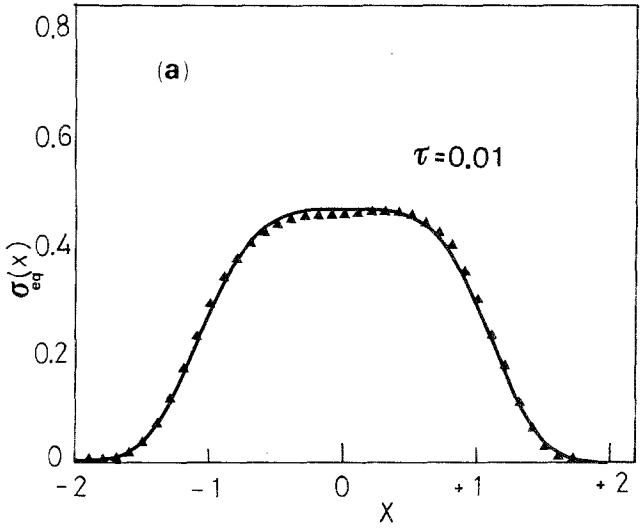
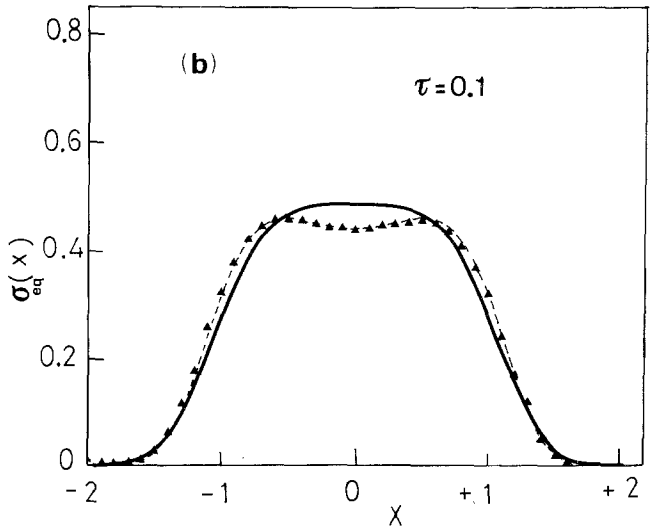


Fig. 4. Equilibrium distribution for the case of the sinusoidal potential of Section 3. (—) The result of the global linearization.[7] (- -) The result obtained by using the LL approximation. (●) The results obtained by using the expression for $\Phi(x)$ of Eq. (3.12), i.e., the BFPA; (○) the results of digital simulation. $D = 0.25$, $k = 1$, and $\omega_0 = 1$.



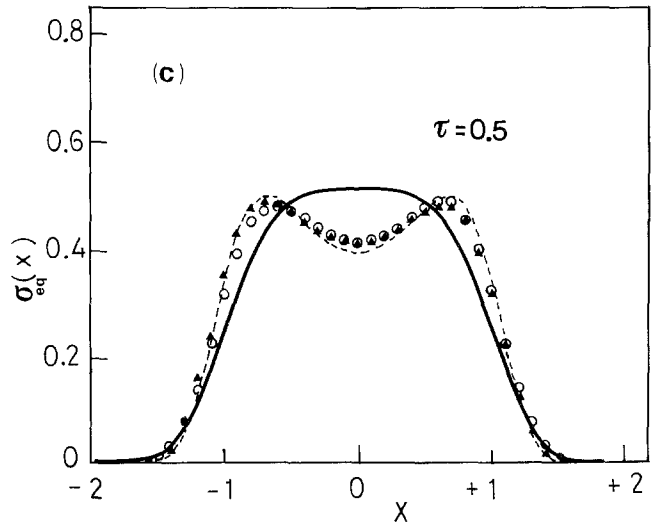


Fig. 4 (*continued*)

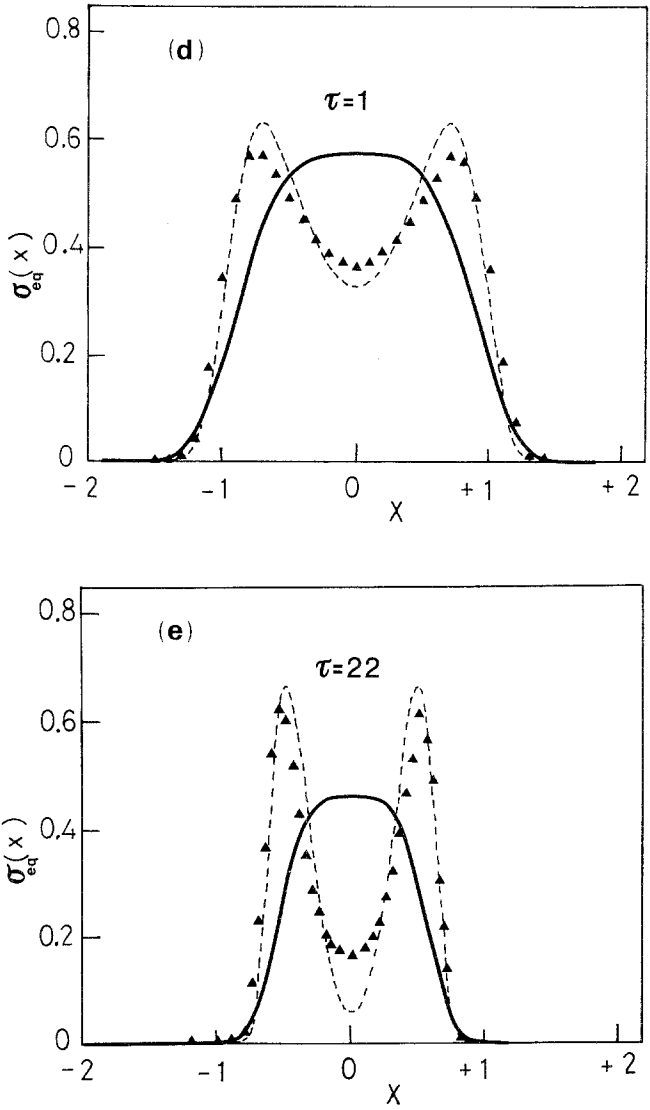


Fig. 4 (*continued*)

7. CONCLUDING REMARKS

This paper proves that the LL approximation is a satisfactory way of getting rid of the terms of higher order in $\mathscr{L}_1$ that provoke the breakdown of the Fokker–Planck structure and invalidate the BFPA at the order in τ higher than the first. For this approximation to be valid a weak diffusion coefficient is required. The LL approximation is much more accurate than the assumption of global linearization,[7] which is a rough assumption totally missing the rich variety of effects provoked by the color of noise. The only seemingly remarkable property of the global linearization concerns the prediction that $\sigma_{eq}(0)$ is finite for any finite value of τ, in accordance with the results of analog and digital simulation. We see from Fig. 2c that at the critical value $\tau = 1/\alpha$ the LL approximation results in a vanishing value of $\sigma_{eq}(0)$, whereas both glogal linearization and experiment result in a finite value of $\sigma_{eq}(0)$. Actually, the wrong prediction of the LL approximation in this case depends on the fact that $\Phi(x) = 1/[1 - \tau\varphi'(x)]$ for $\tau = 1/\alpha$ diverges at $x = 0$. At $\tau < 1/\alpha$, a region with negative diffusion coefficient begins to appear around $x = 0$ and its size becomes larger and larger upon increase of τ, until it covers the whole interval $-(\alpha/3\beta)^{1/2} \leqslant x \leqslant (\alpha/3\beta)^{1/2}$ at $\tau = \infty$.

Actually, this is not an artefact of the LL approximation. To make this aspect clear, note that at the second order in $\mathscr{L}_1$, Eq. (4.1) must be replaced by[25,26]

$$\frac{\partial}{\partial t}\sigma(x;t) = \left\{ -\frac{\partial}{\partial x}\varphi(x) + \gamma D \int_0^t \frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\, e^{-[\gamma - \Pi(x)]s}\, ds \right\} \sigma(x;t) \tag{7.1}$$

In general, this equation should be correct by the terms of higher order in $\mathscr{L}_1$. However, if D is small enough, the LL approximation, $\Pi^{(1)'} = 0$, holds and the terms of higher order in $\mathscr{L}_1$ vanish. On the other hand, the LL approximation obliges us to replace Eq. (7.1) with

$$\frac{\partial}{\partial t}\sigma(x;t) = \left\{ -\frac{\partial}{\partial x}\varphi(x) + \gamma D \int_0^t \frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\, e^{-[\gamma - \Pi^{(1)}(x)]s}\, ds \right\} \sigma(x;t) \tag{7.2}$$

which is not affected by any pathology. Due to the fact that $\Pi^{(1)}(x)$ is a standard function of x, the exponential $\exp[\Pi^{(1)}(x)s]$ turns out to be always a positive number {the same property is not shared in general by the formal exponential $\exp[\Pi(x)s]$}.

The pathological aspects above are consequences of the improper replacement of the upper limit of time integration t on the rhs of Eq. (7.2)

with infinity. Numerical integration of Eq. (7.2) in the critical correlation time region $\tau > 1/\alpha$ [$\psi(x) = 1$, $\varphi(x) = \alpha x - \beta x^3$], where the diffusion coefficient around $x = 0$ tends to diverge for $t \to \infty$ and the stationary assumption $(\partial/\partial t)\,\sigma(x; t) = 0$ no longer holds, shows[26] that Eq. (7.2) produces correct predictions through the whole interval $0 \leqslant \tau \leqslant \infty$.

APPENDIX

A1. Digital Simulation

The computer simulation virtually used the same algorithm as that of ref. 11. The major difference is the Gaussian noise generator: due to reasons of speed, the algorithm used is the one described in ref. 27 (routine NORRAN of CERNLIB). The routine is believed to be about five times faster than the Box–Mueller formula of ref. 11.

Special attention has been devoted to the simulation of the sine potential [we fix in Eq. (2.34) $\omega_0 = 1$ and $k = 1$]. The "particle" performing the random motion was allowed to diffuse from well to well: it is generally accepted that the simulation might become less accurate if the random particle is allowed to evolve very far from the central well (in our case the well centered in $\pi/2$). In the present situation, however, due to the low value of the diffusion coefficient herein considered in order to obtain the result of the next section, even letting the particle move freely, in no case during the course of each simulation did it move farther than two wells away from the one in $\pi/2$. The rare events of a particle escaping from a well into another one are used to fix a new initial condition. In other words, if the particle escapes from a well centered at $x = \pi/2$ into a well centered at $x = 3\pi/2$, this is equivalent to considering a new trajectory with the particle injected at $x = 0$. This expedient serves the purpose of avoiding the need for the particle to reenter the tagged well from the side opposite to that of escape, thereby allowing the speed of our computer experiment to increase. In the simulation the time step was 10^{-3}. Upon changing the initial seed to initialize the random generator, no significan change was obtained for the equilibrium distributions.

A2. Analog Simulation

The analog simulation of Eq. (1.1) with $\varphi(x) = \alpha x - \beta x^3$ ($\alpha > 0$) and $\varphi(x) = -\beta x^3$ was obtained using a minimum components technique.[28] In Fig. 5 we show the scheme of the simulation electric circuit. An operational amplifier is coupled with two Analog Device multipliers (AD534). The output of a noise generator was sent to a low-pass filter before being applied

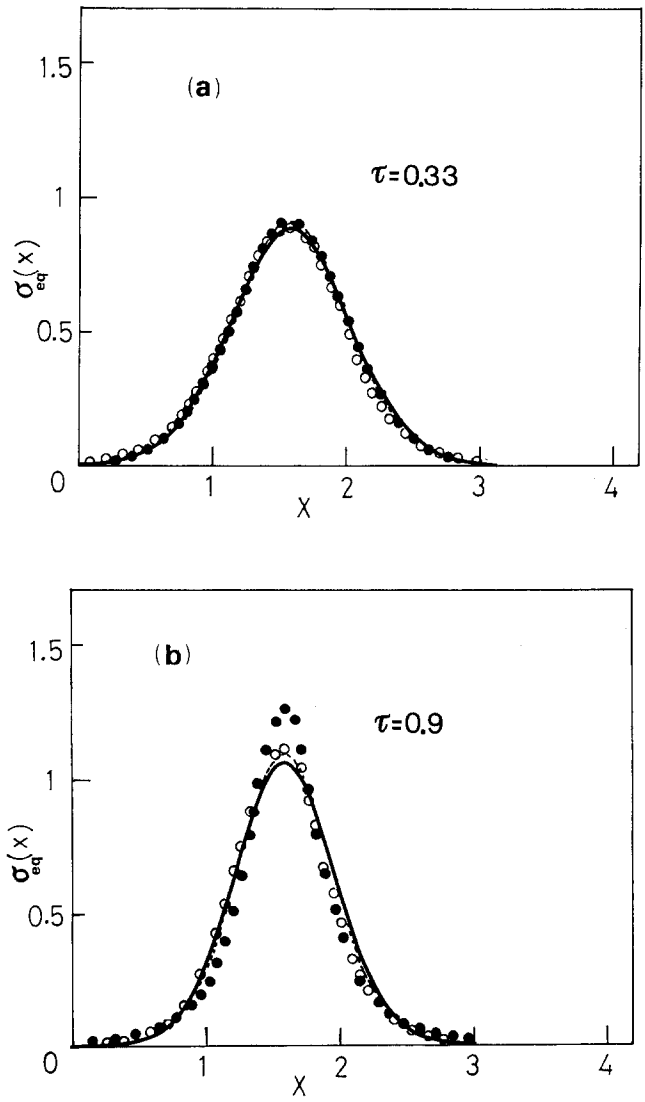


Fig. 5. Scheme of the electric circuit used to simulate the cases $\alpha > 0$ ($V_0 = 1$) and $\alpha = 0$ ($V_0 = 0$). This scheme shows how to relate the corresponding stochastic differential equations to the parameters of the circuit. $R = 10\ \text{k}\Omega$, $R_i = 50$–$100\ \text{k}\Omega$, and $C = 4.7\ \text{nF}$.

to the input of the analog device. The cutoff frequency of the filter gives the value of the time correlation of the noise. Time scaling is necessary so as to refer to the parameters of the case $\varphi(x)=\alpha x-\beta x^3$ $(\alpha>0)$ and $\varphi(x)=-\beta x^3$. More details can be found in ref. 28. We send the output of the differential operational to a computer so as to get the statistical distribution $\sigma_{eq}(x)$. Our device gives the simulation of the bistable system $(\alpha>0$ or the "x^4 potential" system $(\alpha=0)$ by changing the voltage V_0 applied to the circuit (see the equation in Fig. 5). Thus, by putting V_0 close to zero we get the x^4 potential; otherwise we put $V_0=1$ for the bistable potential.

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